Cooper pairs and exclusion statistics from coupled free-fermion chains

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January 29, 2007

Abstract

We show how to couple two free-fermion chains so that the excitations consist of Cooper pairs with zero energy, and free particles obeying (mutual) exclusion statistics. This behavior is reminiscent of anyonic superconductivity, and of a ferromagnetic version of the Haldane-Shastry spin chain, although here the interactions are local. We solve this model using the nested Bethe ansatz, and find all the eigenstates; the Cooper pairs correspond to exact-string or "0/0" solutions of the Bethe equations. We show how the model possesses an infinite-dimensional symmetry algebra, which is a supersymmetric version of the Yangian symmetry algebra for the Haldane-Shastry model.

1 Introduction

The Bethe ansatz is one of the few reliable methods physicists have for probing properties of systems with strong correlations. For the systems to which it is applicable, it gives a way of doing exact computations.

One of the most valuable insights it gives is in the quasiparticle spectrum of one-dimensional quantum systems. A number of profound but highly non-obvious properties have been discovered using the Bethe ansatz. For example, Bethe's original paper gives the tools necessary to show that the quasiparticles in the one-dimensional antiferromagnetic Heisenberg model are gapless and have a linear dispersion relation [1]. It took more than half a century and some novel non-perturbative physics to understand that these excitations have spin 1/2, and to explain this in terms of field theory [2, 3]. After the Bethe ansatz was generalized to systems with multiple types of particles via the invention of the nested Bethe ansatz [4], it was subsequently shown how the spectrum of the one-dimensional Hubbard model is spin-charge separated. Even though the Hubbard model is comprised of electrons with both charge and spin, the quasiparticles in one dimension have either spin or charge, but not both [5]. It was again quite some time before this result was understood in depth from the field-theory point-of-view.

In this paper we use the Bethe ansatz to find and describe a strongly-correlated one-dimensional system with two kinds of interesting excitations. The system consists of two coupled free-fermion chains; the excitations are Cooper pairs and exclusons.

Cooper pairs of course are familiar from studies of superconductivity and fermionic superfluidity. They are comprised of two fermions bound (in momentum space) into a state which has zero energy. The Cooper pairs here arise in a fashion very reminiscent of anyonic superconductivity [6]. Anyons are quasiparticles in two dimensions which have statistics generalizing those of fermions and bosons: they pick up a phase under exchange, instead of a sign. One kind of interaction between our fermions is a one-dimensional analog: a fermion on one chain picks up a phase $\pm i$ when it moves past a fermion on the other chain.

Exclusons are also a generalization of the idea of fermions and bosons [7]. However, as opposed to anyons, they can occur in one (or any) dimension. The usual notion of statistics is not applicable in one dimension because particles can not be exchanged without coming close to each other. Nevertheless, in any dimension fermions can be defined as particles which are allowed at most one to a level, and bosons those where any number is allowed. Exclusion statistics describe more general rules for how particles fill energy levels. In our model, we will show that there are two types of exclusons, associated with each of the two chains. Each of the two types has their own (identical) set of energy levels, which with appropriate boundary conditions are those of free particles. Amongst themselves, each type behaves like a fermion, allowing at most one quasiparticle per level. However, when say an excluson of type 1 is in a given energy level, not only does it forbid another of type 1 from having that energy, but it also forbids a particle of type 2 from having that energy as well.

Exclusons in one dimension are well known to occur as quasiparticle excitations of the Haldane-Shastry chain, which is a variant of the Heisenberg model with long-range interactions [8]. In both our model and the Haldane-Shastry chain [9], a nice property is that the interaction between exclusons is purely statistical. In other words, the energy levels are those of free particles: the only interactions are in the rules for filling these levels. It is important to note that it is the quasiparticles which have only exclusion interactions: the original degrees of freedom in both cases are strongly interacting. One very interesting distinction between our model and the Haldane-Shastry chain is that the interactions here are nearest-neighbor.

Even after 75 years, computations using the Bethe ansatz are usually quite technical, and this one is no exception. However, we find it remarkable that here the technical complications result in elegant physics. For example, to obtain our results we impose open boundary conditions, which require much more effort to deal with in the Bethe ansatz than periodic ones do. Nevertheless, not only does the Bethe ansatz still work here, but the fact eigenstates are standing waves ends up making it possible to find them all in closed form. This is how we manage to show that this strongly-interacting system has quasiparticle excitations which are free exclusons and Cooper pairs.

The Cooper pairs themselves arise as a consequence of resolving another technical complication: they turn out to be "exact-string" solutions of the Bethe equations. The Bethe equations are coupled polynomial equations whose solutions determine the allowed momenta in the wavefunction. String solutions are those for which some of these polynomials vanish in the thermodynamic limit; they occur even in the Heisenberg model discussed in Bethe's original paper. In an exact-string solution, they vanish even for a finite number of sites. In the way the Bethe equations are usually written, this is a singular "0/0" limit: both the numerator and the denominator of a ratio are exactly zero. However, it is explained in detail in [10] how to define the Bethe equations so that exact-string solutions are not singular. Including such solutions is necessary to ensure that all eigenstates of the Hamiltonian follow from the Bethe ansatz.

Moreover, the presence of exact-string solutions in the XXZ spin chain described in [10] is closely related to the appearance of a very interesting extended-symmetry algebra there [11].

We find an analogous algebra for our model as well. Because our model turns out to have a supersymmetry, this extended symmetry algebra is an extension of the supersymmetry algebra. A similar symmetry algebra called the Yangian appears in the Haldane-Shastry chain [12].

In section 2, we present our model and describe in detail the results of this paper. In section 3, we derive these results using the Bethe ansatz. In section 4, we discuss the extended symmetry algebra, and its connection to supersymmetry. We present our conclusions in section 5.

2 The model and the results

We consider two free-fermion chains, each with L sites. It is convenient to label the sites on one chain by odd integers $1, 3, \ldots, 2L-1$ and on the other by even integers $2, 4, \ldots, 2L$. The spinless fermions on the two chains are created by the operators $c_j^\dagger$ which anticommute except for $\{c_k, c_j^\dagger\} = \delta_{jk}$. The fermions on a given chain do not interact among themselves, except of course for the requirement that two fermions cannot occupy the same site. The number operator $n_j = c_j^\dagger c_j$ is 1 if the site j is occupied and 0 if it is unoccupied. We allow hopping along each chain, but not from one chain to the other, so the fermion-number operators

$$F_1 = \sum_{k=1}^{L} n_{2k-1} \; , \qquad F_2 = \sum_{k=1}^{L} n_{2k}$$

are each conserved. A state with fermion numbers f_1 and f_2 on the chains is of the form

$$\sum_{j_1 \ldots j_{f_1+f_2}} \varphi\left(j_1, \ldots, j_{f_1} | j_{f_1+1}, \ldots j_{f_1+f_2}\right) c_{j_1}^\dagger \ldots c_{j_{f1+f2}}^\dagger |0\rangle \; , \tag{1}$$

where $|0\rangle$ is the empty state. Here and for the remainder of the paper, j_a is odd if $a \leq f_1$ and even if $a > f_1$.

Let us first consider two decoupled free-fermion chains with open boundary conditions. We include hopping, a specific chemical potential, and a boundary magnetic field, so that the Hamiltonian is

$$H_f = 2F_1 + 2F_2 + \sum_{j=2}^{2L-1} \left[c_{j+1}^\dagger c_{j-1} + c_{j-1}^\dagger c_{j+1} \right] + H_b \; . \tag{2}$$

The specific boundary magnetic field we choose gives

$$H_b = -n_1 - n_{2L} \; . \tag{3}$$

so that it lowers the energy when a particle is on one end of each chain.

An elementary computation gives the eigenstates of H_f to be of the form (1) with

$$\varphi_{\text{free}} = \prod_{a=1}^{f_1+f_2} \sin(p_a j_a) \tag{4}$$

where

$$p_a = m_a \frac{\pi}{2L+1} \tag{5}$$

with m_a an integer from $1 \ldots L$. Even though the open boundary conditions mean that the system is not translation-invariant and there is no overall momentum conservation, we still refer

to the p_a as the momenta. The shift in the denominator of (5) is due to the boundary magnetic field. The energy of this eigenstate is

$$E = \sum_{a=1}^{f_1+f_2} (2 + 2\cos(2p_a)). \tag{6}$$

To impose Fermi statistics and forbid double occupancy, one sums over these eigenstates as in (12) below. One can have $m_a = m_b$ only if $a = b$ or if a and b label particles on different chains, i.e. $a \le f_1$ and $b > f_1$. For a given f_1 and f_2 there are therefore $\binom{L}{f_1}\binom{L}{f_2}$ different choices of momentum. This is the number of states, so (4) is a complete set of states.

We now couple the two chains in a zig-zag fashion, so that each site j is adjacent to the sites $j \pm 1$ on the other chain. There are two kinds of interaction between the chains. The first is simply an attractive interaction between nearest-neighbor fermions on different chains. The second is a phase factor $\pm i$ picked up when a fermion on one chain hops past one on the other, which can be thought of as resulting from flux tubes attached to the particles. The Hamiltonian is then $H_f + H_c$, where

$$H_c = -\sum_{j=2}^{2L-1} \left[(1-i)c^\dagger_{j+1} n_j c_{j-1} + (1+i)c^\dagger_{j-1} n_j c_{j+1}\right] \; - \; 2\sum_{j=1}^{2L-1} n_j n_{j+1} \; . \tag{7}$$

The specific values of the couplings in (2) and (7) are chosen because they result in a model with a great deal of symmetry, which we will discuss in depth in this paper. Some of the symmetries (e.g. integrability) persist under certain deformations of the couplings, but we will confine our studies to this specific Hamiltonian.

The model with Hamiltonian $H_f + H_c$ is solvable using the nested Bethe ansatz, and in the next section we present the solution in detail. The Bethe equations can be solved explicitly, and the resulting physics is quite elegant. We therefore discuss the key physical results here.

We find that in this strongly-coupled theory, the energy levels are still those of free fermions, i.e. they are given by (6) with momenta quantized in the free-fermion form (5). However, the theory is *not* the same as that of the decoupled chains: the strong interactions have several interesting effects. The wavefunctions can still be written in a form quite similar to (4), so that each state can be characterized by a set of momenta p_a with $a = 1, \dots, f_1 + f_2$. The energy is still given by (6). However, not all momenta obey (5) or are even real. A pair of particles, one from each chain, can form a *Cooper pair*. Namely, two of the momenta (say p_1 and p_2) obey

$$\cos^2(p_1) = -\cos^2(p_2). \tag{8}$$

This existence of Cooper pairs does not contradict the first sentence of this paragraph, because each pair obeying (8) has no net contribution to the energy. If $f_1 = f_2$, then one can pair all the particles and obtain a state with energy zero. We show on general grounds in section 4 that $E \ge 0$, so a state comprised entirely of Cooper pairs is a ground state.

Any momenta not part of a Cooper pair must obey (5), so the energy levels are indeed those of free fermions. The degeneracy of each level, however, is not the same as that for the decoupled chains. One can add Cooper pairs without changing the energy, which obviously will change the degeneracies. Even in a state with no Cooper pairs, there is a crucial difference. In both coupled and decoupled chains, the numbers f_1 and f_2 of fermions on each chain are conserved. When the chains are decoupled, two momenta can be the same, if they are associated with particles on different chains. When the chains are coupled via H_c, we find that *no two momenta can be equal.*

In fact, the individual momenta cannot be associated with one chain or the other like they can in the decoupled case: for C Cooper pairs, the wavefunction is labeled by $f_1 + f_2 - 2C$ distinct momenta obeying (5), and C pairs of momenta obeying (8).

Thus we can interpret the states with non-zero energy in terms of quasiparticles with momentum (5). We call these quasiparticles exclusons for the following reason. Consider fixed numbers f_1 and f_2 of fermions on each chain, and also a fixed number of Cooper pairs C. There are two types of exclusons: $N_1 = f_1 - C$ of type 1, and $N_2 = f_2 - C$ of type 2. The energy of this state is given by

$$E = \sum_{a=1}^{N_1+N_2} 4\cos^2(m_a\pi/(2L+1)) \tag{9}$$

with the $m_a \in \{1, \dots L\}$ distinct integers. We show in section 3 that the number of states with this energy is

$$d_E = \binom{N_1+N_2}{N_1}\binom{L-N_1-N_2}{C} \tag{10}$$

We have checked this formula for the degeneracy directly by numerically diagonalizing the Hamiltonian (which of course is how we originally discovered it).

In the quasiparticle interpretation of (9,10), the first binomial arises from the exclusons, the second from the Cooper pairs. The N_i exclusons of each type are indistinguishable from each other, so the number of ways of assigning $N_1 + N_2$ momenta to them which result in distinct wavefunctions is $\binom{N_1+N_2}{N_1}$. The interpretation of the second piece is that the Cooper pairs can be thought of indistinguishable quasiparticles like the exclusons, and there is one possible Cooper pair for each allowed momentum. Moreover, the presence of an excluson with a given momentum prohibits a Cooper pair from occupying the corresponding level.

The exclusons, the quasiparticles of non-zero energy, obey mutual *exclusion statistics* [7]. Even though in our counting we have treated exclusons on chain 1 and on chain 2 as distinct species, only one can fill a given momentum level. This is thus a generalized exclusion principle: the presence of one kind of quasiparticle affects how others fill levels. A mutual exclusion principle is typical for systems solvable by the nested Bethe ansatz; what is novel here is that except for this mutual exclusion, the quasiparticles are otherwise blind to each other's presence. Namely, the energy levels are unaffected by how many quasiparticles are present: the only effects are on how the levels fill.

This mutual exclusion principle involves the Cooper pairs as well. Fermions of the same type must still of course obey Pauli exclusion. Thus the presence of exclusons restricts the allowed momenta of the Cooper pairs, and reduces the size of the space of states for the Cooper pairs as well.

One important issue to note is that the momentum of one of the particles in a Cooper pair is arbitrary, as long as it is not the same as any of the exclusons. The arbitrariness is due to the degeneracy: any linear combination of the d_E states with fixed excluson momenta is an eigenstate of the Hamiltonian. This arbitrariness in choosing certain momenta in the Bethe ansatz is discussed in depth in [10], and we will return to this issue in section 3. The counting rules become clearer if we choose any arbitrary momenta to be one of the values obeying (5) which are not already present. The momentum of the second particle in each Cooper pair is then fixed by (8). The second factor in the degeneracy (10) then simply corresponds to the number of different such choices.

The factor d_E looks somewhat complicated. However, it ends up yielding an extremely simple and elegant formula for the *grand* canonical partition function. The allowed momentum values

(5) are independent of how many quasiparticles are present, so we can treat the grand canonical partition function as a product over momentum levels in the usual free-particle fashion. The reason this is possible here is that the way to fix the arbitrariness in the Cooper-pair momenta uniquely is to make it obey the same rules as the exclusons do. In other words, there are four possibilities for each of these momentum levels: it is empty, occupied by an excluson on chain 1 or by an excluson on chain 2, or occupied by a Cooper pair. The first and last of these possibilities have energy 0, whereas an excluson in the ath momentum level has energy $4\cos^2(p_a)$. If λ_1 and λ_2 are the fugacities for each fermion on chains 1 and 2 (not including the chemical potentials of -2 already in (2)), the grand canonical partition function at temperature T is then

$$Z = \prod_{m=1}^{L} \left[1 + \lambda_C + (\lambda_1 + \lambda_2) e^{-4\cos^2(m\pi/(2L+1))/T}\right] \tag{11}$$

where $\lambda_C = \lambda_1\lambda_2$. In the limit $T \to \infty$, Z simply counts the number of allowed states. This limit indeed yields the correct $Z \to 4^L = 2^{2L}$ for two chains of length L (each site is either occupied or unoccupied). From (11) one can compute any thermodynamic quantity, such as the specific heat, or the density of Cooper pairs.

One can easily recover the degeneracies in (10) by expanding out the product in Z, i.e.

$$Z = \sum_{\mathcal{N}=0}^{L} \sum_{N_1=0}^{\mathcal{N}} \sum_{C=0}^{L-\mathcal{N}} \sum_{\{m_a\}} d_E\,(\lambda_C)^C (\lambda_1)^{N_1} (\lambda_2)^{N_2} e^{-E/T}$$

where $\mathcal{N} = N_1 + N_2$ is the total number of exclusons, E is given in (9) and the sum $\sum_{\{m_a\}}$ is over sets of integers obeying $1 \le m_1 < m_2 < \cdots < m_{\mathcal{N}} \le L$. Note that the maximum number of exclusons is L, so if $f_1 + f_2 > L$, then $C \ge f_1 + f_2 - L$.

One can find a Hamiltonian with the same grand canonical partition function by considering spin-1/2 fermions with long-range interactions. Having two fermions of different spin and the same momentum corresponds to a Cooper pair. To make the Cooper pair have energy zero, the Hamiltonian must include a binding energy for fermions of the same momentum. Such a Hamiltonian is simple to write in momentum space, but when Fourier-transforming back to position space, one obtains a complicated long-range interaction.

Before proceeding to the solution by Bethe ansatz we comment once again on the fractional statistics aspects of the model. We already observed that particles on a given chain act as fermions, while moving a fermion on one chain past one on the other involves phase factors of $\pm i$. This then suggests a statistics matrix (in the sense of Haldane [7]) $G = \{g_{ij}\}$ with $g_{11} = g_{22} = 1$, $g_{12} = g_{21} = 1/2$ for these 'bare' particles. For the exclusons, which fully exclude one another from a given level, the natural assignment would be $g_{11} = g_{22} = 1$, $g_{12} = g_{21} = 1$. We observe though that the 1-level partition sum that is associated with this choice of G (using results of [13, 14]) is different from the simple result that we find here.

3 The solution using the nested Bethe ansatz

In this section we derive all the claims made in the last section by using the Bethe ansatz.

When all fermions are on the same chain, there are no interactions save forbidding double-occupancy. Exact eigenstates are given in (4) above. To forbid double occupancy and anti-symmetrize them appropriately, we introduce a permutation $P = (P1, P2, \ldots, Pf)$ of the integers

$(1, 2, \ldots, f)$, where f is the total number of fermions. Then we have

$$\varphi_{(f,0)} = \sum_P (-1)^{|P|} \prod_{a=1}^{f} \sin(p_{Pa} j_a) \; . \qquad (12)$$

where $|P|$ is the order of the permutation, and p_a satisfies (5).

The Bethe ansatz is a very natural generalization of (12). In situations like ours where there is more than one species of particle, imposing Fermi statistics does not automatically fix the relative coefficients between different orderings. One must therefore use the *nested Bethe ansatz* [4]. It is thus necessary to label the orderings by another permutation Q of $(1, 2, \ldots f)$, so that φ^Q is the part of the wavefunction where $j_{Q1} < j_{Q2} < \cdots < j_{Qf}$. It is simplest to first work with periodic instead of open boundary conditions. This amounts to changing the sums in H_f in (2) and H_c in (7) to run from 1 to L, interpreting positions mod $2L$, and setting $H_b = 0$. The Bethe ansatz for the eigenstate with periodic boundary conditions is then

$$\varphi^Q = \sum_P A_P^Q \exp\left(i \sum_{a=1}^{f} p_{Pa} j_{Qa} \right) . \qquad (13)$$

For open boundary conditions, the eigenstates consist of sums over these φ^Q with $\pm p_a$, i.e. are standing waves. Note that the momenta are permuted over all $f = f_1 + f_2$ fermions.

The miracle of the Bethe ansatz in general is that in some situations, the p_a and the A_P^Q can be found which result in an eigenstate of the Hamiltonian. The corresponding energy is exactly that given in (6). The p_a are given as solutions of what are called the *Bethe equations*. The additional miracle here is that for open boundary conditions with (3), the Bethe equations can be solved explicitly, yielding the results described in section 2.

Let us first discuss the case of two fermions. If both are on the same chain, the only constraint is no double occupancy, i.e. $\varphi^Q(j, j) = 0$ for all Q. This means that

$$A_{12}^Q = -A_{21}^Q \qquad \text{for } f_1 = 2,\ f_2 = 0, \qquad (14)$$

for both $Q = (12), (21)$. The solution is much more interesting when each chain has one fermion. To make the equations look simpler, let $x = e^{ip_1}$ and $y = e^{ip_2}$, and j and k represent the locations of the fermions on chains 1 and 2 respectively; by our conventions j is an odd integer and k even. so we have

$$\begin{aligned} \varphi^{12}(j|k) &= A_{12}^{12} x^j y^k + A_{21}^{12} y^j x^k \qquad && \text{for } j < k \\ \varphi^{21}(j|k) &= A_{12}^{21} x^k y^j + A_{21}^{21} y^k x^j \qquad && \text{for } k < j \; . \end{aligned}$$

We then let the Hamiltonian act on the state (1) with this φ. Ignoring the boundary conditions momentarily, a little algebra shows that this is an eigenstate if

$$2\varphi^Q(k-1|k) = -\varphi^Q(k+1|k) - i\varphi^{Q'}(k+1|k) - \varphi^Q(k-1|k-2) + i\varphi^{Q'}(k-1|k-2) \qquad (15)$$

and its parity conjugate

$$2\varphi^{Q'}(k+1|k) = -\varphi^{Q'}(k+1|k+2) - i\varphi^Q(k+1|k+2) - \varphi^{Q'}(k-1|k) + i\varphi^Q(k-1|k) \qquad (16)$$

for all even k. Here $Q = 12$ and $Q' = 21$, but it is straightforward to show that these constraints apply in general when Q' is defined by reversing Qa and $Q(a{+}1)$ in Q, i.e. $Q' = (\ldots Q(a{-}1), Q(a{+}$

$1), Qa, Q(a+2), \dots)$. Then (15) and (16) must hold whenever $Qa \le f_1$ and $Q(a+1) > f_1$, i.e. j_{Qa} is odd (is on chain 1), and $j_{Q(a+1)}$ is even (is on chain 2).

The eigenvalue for such an eigenvector is simply (6), i.e.

$$E_{(1,1)} = 4 + x^2 + x^{-2} + y^2 + y^{-2} = (x + x^{-1})^2 + (y + y^{-1})^2. \tag{17}$$

Plugging the Bethe ansatz (13) for the eigenvector into the constraints (15,16) gives two equations for the four unknowns A_P^Q. We find

$$E_{(1,1)} \begin{pmatrix} A_{21}^{12} \\ A_{21}^{21} \end{pmatrix} = \begin{pmatrix} -2(x+x^{-1})(y+y^{-1}) & i((x+x^{-1})^2 - (y+y^{-1})^2) \\ i((x+x^{-1})^2 - (y+y^{-1})^2) & -2(x+x^{-1})(y+y^{-1}) \end{pmatrix} \begin{pmatrix} A_{12}^{12} \\ A_{12}^{21} \end{pmatrix} \tag{18}$$

When $(y + y^{-1}) = \pm i(x + x^{-1})$, we have $E_{(1,1)} = 0$, and the particles form a Cooper pair. Since the matrix has determinant $(E_{(1,1)})^2$, it is still possible to solve (18) for a Cooper pair, yielding $A_{12}^{12} = \pm A_{12}^{21}$ and $A_{21}^{12} = A_{21}^{21} = 0$.

This eigenstate must also satisfy the boundary conditions. For periodic untwisted boundary conditions, they are simply

$$(xy)^L = 1, \qquad y^L A_{12}^{12} = A_{21}^{21}, \qquad x^L A_{12}^{21} = A_{12}^{21}\ .$$

The first of these is simply the requirement that the total momentum be an integer multiple of $2\pi/L$, while the others are the individual momentum quantization conditions in the presence of interactions. These equations can be solved to yield all of the eigenstates. The Cooper pairs do not occur here, since to satisfy both the periodic boundary conditions and (18) would require all $A_P^Q = 0$.

The generalization of this computation to arbitrary numbers of particles is now standard, even though it is a mere 39 years since the invention of the nested Bethe ansatz. The key ingredient is the R-matrix, which encodes the relations on the A_P^Q necessary to make the state an eigenvector. These relations are given in (14,15,16): one must satisfy the constraints for each pair of fermions when they become adjacent. Let $P = (\dots Pa, P(a+1)\dots)$ and $P' = (\dots P(a+1), Pa\dots)$. Then we have

$$E_{Pa,P(a+1)} A_{P',Q} = R(p_{Pa}, p_{P(a+1)}) A_{P,Q} \tag{19}$$

where

$$R(p_a, p_b) = \begin{pmatrix} -E_{ab} & 0 & 0 & 0 \\ 0 & -8\cos(p_a)\cos(p_b) & 4i(\cos^2(p_a) - \cos^2(p_b)) & 0 \\ 0 & 4i(\cos^2(p_a) - \cos^2(p_b)) & -8\cos(p_a)\cos(p_b) & 0 \\ 0 & 0 & 0 & -E_{ab} \end{pmatrix} \tag{20}$$

where E_{ab} is the energy of particles with momenta p_a and p_b, i.e.

$$E_{ab} = 4(\cos^2(p_a) + \cos^2(p_b)).$$

The R-matrix is 4×4 because there are four possibilities for which chains the fermions at locations j_{Qa} and $j_{Q(a+1)}$ occupy. The 2×2 block in the middle applies to the cases where the fermions are on different chains, and is the same as that in (18). The inverse $(R(p_a, p_b))^{-1} = R(p_b, p_a)/(E_{ab})^2$, as necessary for consistency. We have followed [10] and written (19) so that it still is applicable when $E_{ab} \to 0$, i.e. when p_a and p_b satisfy the Cooper pair condition (8).

The R matrix relates coefficients $A_{P'}^Q$ to A_P^Q. Repeatedly applying (19) gives all the A_P^Q in terms of those for a given permutation, which we label as $P = 0$. For periodic boundary

conditions the A_0^Q are then related to each other by imposing the boundary conditions as well. To derive the resulting Bethe equations in situations like ours where the R matrix is non-diagonal, one must do a second Bethe ansatz, the "nested" part of the nested Bethe ansatz [4]. Here this requires one level of nesting, and we omit it here because we will not need it. A vital consistency condition in applying (19) is that the R-matrix satisfy the Yang-Baxter equation, which ensures that this procedure defines all the A_P^Q consistently. This R-matrix (20) indeed satisfies the Yang-Baxter equation, and for example arises in vertex models [15] and the $t-J$ model with the $SU(2)$ symmetry deformed to $SU(2)_q$ [16]. It also arises in the scattering matrix of the kinks in the sine-Gordon model at its supersymmetric point $\beta^2 = 16\pi/3$ [17]. We note also that this R-matrix satisfies a special condition called the "free-fermion" condition [18], which is part of the reason behind the miracles in this paper.

We now return to the boundary conditions of interest in this paper: open ones with a boundary magnetic field (3). To implement these open boundary conditions, we need to take combinations of wavefunctions with $\pm p_a$. The Bethe ansatz (13) is generalized here to

$$\varphi^Q = \sum_P A_P^Q \prod_{a=1}^{f} \left(e^{ip_{Pa} j_{Qa}} - \gamma_{Pa}^Q e^{-ip_{Pa} j_{Qa}} \right). \tag{21}$$

The boundary conditions can be satisfied by choosing the coefficients γ_{Pa}^Q for the momentum corresponding to the leftmost and rightmost fermions on each chain. By definition of Q, the leftmost fermion is located at j_{Q1}, while the rightmost one is located at j_{Qf}. For each Q, the form of the boundary conditions depends on which chain the leftmost and rightmost fermions are on. We have

$$0 = \begin{cases} \varphi^Q(j_{Q1} = 1, \dots | \dots) + \varphi^Q(j_{Q1} = -1, \dots | \dots) & \text{if } Q1 \le f_1 \\ \varphi^Q(\dots | j_{Q1} = 0, \dots) & \text{if } Q1 > f_1 \end{cases}$$

$$0 = \begin{cases} \varphi^Q(\dots | \dots, j_{Qf} = 2L) + \varphi^Q(\dots | \dots, j_{Qf} = 2L+2) & \text{if } Qf > f_1 \\ \varphi^Q(\dots, j_{Qf} = 2L+1 | \dots) & \text{if } Qf \le f_1 \end{cases}$$

Plugging (21) into these boundary conditions, we find the same constraint for either condition. We have for the left end

$$\gamma_{P1}^Q = -1 \tag{22}$$

for all Q. Likewise, whichever boundary condition for the right end is applicable, we find

$$e^{-ip_{Pf} 2(2L+1)} \gamma_{Pf}^Q = -1. \tag{23}$$

First let us consider the case where $E_{ab} \neq 0$ for any choice of a, b, i.e. there are no Cooper pairs. Deriving the Bethe equations and the relations among the coefficients A_P^Q now proceeds in the same fashion as for periodic boundary conditions. In fact, the computation is virtually identical because the R-matrix is independent of the signs of p_a and p_b. Thus we can expand out the product in the ansatz (21), and apply the same relation (19) to any of the resulting 2^f terms for a given A_Q^P. Consistency then requires that all the γ_{Pa}^Q for a given Q to be the same. Since one of them is fixed to be -1 by the boundary conditions at the left end, this means $\gamma_{Pa}^Q = -1$ for all P, a, and Q. We still need to satisfy the boundary conditions (23) at the right end. Since these must hold for all permutations P and Q, the only way this is possible is if

$$e^{ip_a 2(2L+1)} = 1 \tag{24}$$

for all a. This is precisely the momentum quantization condition for a free fermion

$$p_a = m_a \frac{\pi}{2L+1}$$

given in (5)!

Another miracle is that there are multiple states with the same energy: fixing the p_a does *not* fix all the A_P^Q. Because the boundary conditions are the same for all Q, they do not restrict the A_P^Q at all. The relation (19) can be used to relate the A_P^Q to those for a given permutation. Labeling this given permutation by $P = 0$, (19) determines the other A_P^Q in terms of A_0^Q, but it does not fix the latter. The only relation between the different A_0^Q is therefore that required by Fermi statistics for the fermions on each chain. Fermi statistics for chain 1 means that when both $Qa \leq f_1$ and $Qb \leq f_1$, we have $A_P^Q = -A_P^{Q'}$, where Q' differs from Q by swapping Qa and Qb, i.e. if $Q = (\ldots, Qa, \ldots, Qb, \ldots)$ then $Q' = (\ldots, Qb, \ldots, Qa, \ldots)$. Fermi statistics for chain 2 means that when both $Qa > f_1$ and $Qb > f_2$, $A_P^Q = -A_P^{Q'}$ as well. For a given set of momenta satisfying (24), there are therefore $\binom{f_1+f_2}{f_1}$ undetermined coefficients in the Bethe ansatz. All of these are perfectly valid eigenstates, and they all have the same energy. We thus recover the degeneracy d_E in (10) in the special case where there are no Cooper pairs.

The natural way of interpreting these results is for the quasiparticles to be the exclusons described in section 2. These quasiparticles have a statistical interaction, because each momentum p_a must be different: the wavefunction vanishes if $p_a = p_b$ for $a \neq b$, because $R(p_a, p_a) = -1$. Moreover, even though fermions cannot hop between chains, the wavefunction is not a product of separate factors for each chain. A given momenta is not associated with one chain or the other: the sum over P includes permutations over all $f = f_1 + f_2$ momenta. These facts are exactly what happens with two species of quasiparticles obeying exclusion statistics.

We now can see explicitly that we must include Cooper pairs to make the Bethe ansatz complete. There are $\binom{L}{f}$ choices of momenta, each with degeneracy $\binom{f}{f_1}$. However,

$$\binom{L}{f_1}\binom{L}{f_2} > \binom{L}{f}\binom{f}{f_1},$$

so there are more states than those involving solely exclusons. The derivation that all momenta must obey the free-particle quantization condition (24) is valid when $E_{ab} \neq 0$ for all p_a and p_b. Therefore the missing states must include at least one Cooper pair, a state where $E_{ab} = 0$ for at least one pair (a, b).

So let us first return to the case where $f_1 = f_2 = 1$, and now let p_1 and p_2 satisfy the Cooper-pair condition (8), so that $E = 0$. As noted above, the requirements (15,16) are satisfied here if $A_{12}^{12} = A_{12}^{21}$, and $A_{21}^{12} = A_{21}^{21} = 0$. The wavefunction of a single Cooper pair is therefore

$$\varphi_C(j|k) = \begin{cases} A(x^j - x^{-j})(y^k - y^{4L+2-k}) & j < k \\ A(x^k - x^{-k})(y^j - y^{4L+2-j}) & k < j \end{cases} \tag{25}$$

where $(y + y^{-1}) = i(x + x^{-1})$. One can check directly that this yields an eigenstate of $H_f + H_c$ with eigenvalue 0 for any value of x. Note that the momentum quantization condition (24) does *not* apply to the momenta in Cooper pairs.

To show that all the missing states are given by Cooper pairs, we count how many linearly independent eigenstates (25) yields. It is useful to define the new variable $z = (x + x^{-1}) = i(y + y^{-1})$. Choosing

$$A = \frac{y^{-(2L+1)}}{(x - x^{-1})(y - y^{-1})}$$

gives φ_C as a polynomial in z^2 (it can be expressed in terms of Chebyshev polynomials explicitly if desired). It is simple to then see that the highest order of z appearing in these polynomials is $z^{2(L-1)}$, and the lowest is z^0. Since z^2 is arbitrary, it follows that the dimension of the space of states of a single Cooper pair is L. This agrees with the general degeneracy formula for $C=1$ and no exclusons, $N_1=N_2=0$. This gives all the missing states for $f_1=f_2=1$: there are $L(L-1)$ two-excluson states, giving the correct total of L^2. Thus we have the nice result that Cooper pairs make up the states missing due to imposing the exclusion statistics.

Now we consider the general case. It is important to note that we do not single out two fermions and call them a Cooper pair. Rather a Cooper pair occurs when two *momenta* obey the condition (8). The relation (19) still is sufficient to satisfy the eigenvector requirements (15,16); (14) is satisfied as long as we divide out any E_{ab} from both sides before taking them to zero.

When Cooper pairs are present, (19) means that some of the A_P^Q vanish, and others must be equal. For C Cooper pairs, we order the p_a so that $\cos(p_{2c-1})=i\cos(p_{2c})$, for $c=1,\dots C$. Given a permutation P, define a_1 and a_2 so that $Pa_1=2c-1$ and $Pa_2=2c$. Then for A_P^Q not to vanish, we find that for all c

1. $(2c-1)$ must appear before $(2c)$ in P, i.e. $a_1<a_2$
2. if there are no fermions between those with the Cooper pair momenta (i.e. $a_2=a_1+1$) then these two particles must be on different chains. This means $Qa_1\le f_1$ and $Qa_2=Q(a_1+1)>f_1$, or vice versa.

In the latter case, the R-matrix relation (19) also requires that $A_P^Q=A_P^{Q'}$, where Q' is Q with a_1 and $a_2=a_1+1$ reversed.

Applying (19) means that the free-particle momentum quantization condition (24) is still applicable for all the momenta not part of Cooper pairs, i.e. p_a for $a>2C$. However the momenta in Cooper pairs need not satisfy it. The reason is as follows. Because $Pa_1=2c-1$ must always appear before $Pa_2=2c$ in a P with a non-vanishing A_P^Q, the fermion at position j_{Qa_1} must always be to the left of a fermion at position j_{Qa_2}. Thus no fermion with momentum p_{2c} can ever be the leftmost particle, and no fermion with momentum p_{2c-1} can ever be rightmost. Therefore, the boundary condition (22) does not apply to any γ_{2c}^Q, and the boundary condition (23) does not apply to any γ_{2c-1}^Q. Since (24) arose from demanding that the two boundary conditions on a given momentum be consistent, it does not apply to any Cooper-pair momentum.

Let us illustrate these conditions with $f_1=2$ and $f_2=1$, where there are 6 different permutations for P and three for Q to consider. (We only need consider (123), (132), and (312) for Q because the other three are given by applying Fermi statistics to the two fermions on chain 1.) With no Cooper pairs, there are three independent A_{123}^Q, as discussed before. For one Cooper pair, we have p_1 and p_2 satisfying (8). This means that when $\widetilde{P}=(213),(231),(321)$, $A_{\widetilde{P}}^Q$ vanishes for all Q. When $P=(123)$, we have $a_1=1$ and $a_2=2$, so this means $A_{123}^{123}=0$ as well. The non-zero ones must obey for example $A_{123}^{132}=A_{123}^{312}$. Thus for the three A_{123}^Q, one is zero and the other two are equal. Applying (19) then determines all the other A_P^Q, and including the boundary conditions requires that p_3 obey the free-fermion condition (5). Therefore all the A_P^Q are given in terms of A_{123}^{312}: all the degeneracies here arise from the arbitrariness of p_1. Counting the different degrees of freedom here is similar to the two-particle case. We replace p_1 and p_2 with z, so that the wavefunction is given in terms of polynomials in z. However, because of the presence of the excluson with momentum p_3, the lowest-order term in φ^{312} is now order z^2, so there are only $L-1$ terms in the polynomial. When there are no other exclusons, there are L terms. The reason for this excluded volume effect is Fermi statistics. The wavefunction must vanish when $j_1=j_2$,

so neither p_2 nor p_1 can be p_3, thus reducing the number of possibilities for the momentum. This yields a total $L(L-1)$ states with one Cooper pair (the factor of L coming from the L possible values of p_3). Likewise, there are $3\binom{L}{3}$ states with no Cooper pairs, the factor of 3 coming from the independent $A^Q_{(123)}$. This is a complete set of states:

$$\binom{L}{1}\binom{L}{2} = 3\binom{L}{3} + L(L-1)$$

To obtain the general degeneracy given in (10), one proceeds in this fashion. The first factor comes from the number of independent A^Q_0, while the second comes from the order of the polynomials in z_c describing the momenta in the cth Cooper pair. Let us first discuss the different A^Q_0, where 0 is our label for $(12\ldots f)$. Applying Fermi statistics to chain 1 relates all the Q obtained by exchanging any of Qa with $Qa \leq f_1$, while applying it to chain 2 relates any obtained by exchanging any of the Qa with $Qa > f_1$. Thus the largest possible number of distinct A^Q_0 is $\binom{f}{f_1}$. We showed above that these are all distinct when $C = 0$. Now consider $C = 1$. The second condition for non-vanishing A^Q_P described above means that A^Q_0 vanishes for any Q where j_{Q1} and j_{Q2} are either both even or both odd. The non-vanishing ones therefore have $Q1 \leq f_1$ and $Q2 > f_1$ or vice-versa. There are $2f_1f_2(f-2)!$ such possibilities. If Q' is Q with $Q1$ and $Q2$ reversed, then $A^Q_0 = A^{Q'}_0$, so the number of distinct possibilities is lowered to $f_1f_2(f-2)!$. Including the effects of Fermi statistics as well means dividing by $f_1!$ and $f_2!$. Thus the total number of distinct A^Q_0 for one Cooper pair is $\binom{f-2}{f_1-1}$. For general C, applying the second condition C times in the analogous fashion gives

$$\frac{f_1!}{(f_1-C)!}\frac{f_2!}{(f_2-C)!}\frac{(f-2C)!}{f_1!f_2!} = \binom{f-2C}{f_1-C}$$

for the number of independent A^Q_0. Thus we indeed recover the first factor in (10).

The second factor likewise follows from essentially the same argument we gave for small f_1 and f_2. The presence of fermions with momenta not obeying the Cooper-pair conditions creates an excluded volume effect reducing the number of independent terms in the polynomials in z_c. The Fermi statistics means that the wavefunction must be symmetric under exchanges of the z_c, implying that the number of linearly-independent choices of the z_c is the binomial

$$\binom{L-(f-2C)}{C}.$$

By writing the grand canonical partition function as at the end of section 2, we saw that including the degeneracy (10) successfully counts all the states in the theory. Thus the Bethe ansatz is complete.

4 The extended (super)symmetry algebra

The extensive degeneracies in the spectrum given by (10) imply that our model must possess an extended symmetry algebra. This is obvious in the quasiparticle basis of states in terms of exclusons and Cooper pairs. A symmetry operator creates or annihilates a Cooper pair, or changes the type of a excluson, preserving the momentum. In this section we describe this symmetry algebra, and how it acts on the orginial fermionic states. The symmetry turns out to be an extension of supersymmetry, and is infinite-dimensional as $L \to \infty$. As we will show, it is quite reminiscent of the Yangian symmetry of the Haldane-Shastry model.

The basic supersymmetry operators act on the exclusons: Q changes type 1 to type 2, and $Q^\dagger$ the inverse. Changing the type of an excluson requires flipping a fermion from one chain to the other. Thus Q hops a particle on chain 1 to chain 2, with $Q^\dagger$ doing the reverse. These operators are non-local: they depend on the locations of the other particles. Let

$$\alpha_k \equiv \sum_{j=1}^{k} (-1)^j n_j .$$

We then define

$$Q = c_2^\dagger c_1 + \sum_{k=1}^{L-1} \left(e^{i\alpha_{2k-1}\pi/2} c_{2k}^\dagger + e^{i\alpha_{2k}\pi/2} c_{2k+2}^\dagger \right) c_{2k+1}. \tag{26}$$

Both Q and $Q^\dagger$ annihilate the Cooper pairs.

This supersymmetry charge commutes with $H = H_f + H_c$ if H_b is the boundary magnetic field in (3). This can be verified directly, but it is more illuminating to work out the entire supersymmetry algebra. Even though Q is written in terms of fermion bilinears, it is effectively fermionic because of the non-local string of operators from the exponential. One finds for its anticommutators

$$Q^2 = 0 \ , \quad (Q^\dagger)^2 = 0 \ , \quad H = \{Q, Q^\dagger\} \ . \tag{27}$$

It then follows from (27) that $[Q, H] = [Q^\dagger, H] = 0$.

One can derive a number of interesting properties of the states directly from the (unextended) supersymmetry algebra (27). Since we have already derived them (and many more) by using the Bethe ansatz, we will just state these properties here; see [19] for detailed explanations of the methods. All eigenvalues E of the Hamiltonian obey $E \geq 0$. States with $E > 0$ form doublets under the supersymmetry algebra, while the $E = 0$ ground states are annihilated by both Q and $Q^\dagger$. The Witten index provides a lower bound on the number of $E = 0$ ground states [20]. Since the supersymmetry generators are fermionic, for purposes of computing the Witten index, each fermion on chain 1 has charge $-1/2$, while each on chain 2 has charge $+1/2$. Then

$$W = \sum_{\text{states}} e^{i(f_2 - f_1)\pi/2} = 2^L,$$

so there are at least 2^L ground states for all possible values of f_1 and f_2 for a given size L. We showed above that a state with $E = 0$ must be comprised solely of Cooper pairs. These occur whenever $f_1 = f_2$, and from (10) we see that the number of such states for a fixed f_1 and f_2 is $\binom{L}{f_1}$ because $C = f_1 = f_2$ here. Thus there are exactly 2^L ground states, and the lower bound from the Witten index is the exact number here. One can find a Hamiltonian with twisted periodic boundary conditions obeying (27). The Witten index is the same, so the Cooper pairs remain for the twisted model, but the other degeneracies no longer occur.

To understand the full degeneracies, supersymmetry alone is not enough. Q does not change the total fermion number, whereas annihilating or creating a Cooper pair will annihilate or create fermions on both chains. Starting from a state with all particles on the upper chain, acting with Q flips one particle to the lower chain. Since Q is nilpotent the action cannot be repeated. Moreover, To obtain the full degeneracies $\binom{f}{f_1}$ of the exclusons one would need f generalized supercharges that can act independently. We have found that a hierarchy of operators Q_p^+, Q_p^-, H_p, with $p = 0, 1, \ldots$, can be constructed with the following algebraic properties

$$\{Q_p^+, Q_q^+\} = 0, \quad \{Q_p^-, Q_q^-\} = 0, \quad \{Q_p^+, Q_q^-\} = H_{p+q} \ , \tag{28}$$

implying that $[Q_p^\pm, H_q] = 0$ for all p, q. We refer to this algebra as a 'super-Yangian', by analogy with the Yangian appearing in the Haldane-Shastry chain [12].

The leading terms in the hierarchy are $Q_0^+ = Q$, $Q_0^- = Q^\dagger$, $H_0 = H_f + H_b + H_c$. For the operator Q_1^+ we choose a form where a particle at site i in the upper chain hops to sites $i+j$, $j = 1, 3, \ldots$ in the lower chain with amplitudes containing an alternating sign $(-1)^{(j-1)/2}$,

$$Q_1^+ = \sum_{k=0}^{L-1} e^{-i\alpha_{2k}\pi/2} \left(\sum_{j=1,3,\ldots,2N-2k-1} (-1)^{(j-1)/2} e^{i\beta_{2k,j}\pi/2} c^\dagger_{2k+1+j} \right) c_{2k+1} \ , \tag{29}$$

with α_{2k} as given above and

$$\beta_{2k,j} \equiv 2(n_{2k+2} + n_{2k+4} + \ldots + n_{2k+j-1}) - (n_{2k+3} + n_{2k+5} + \ldots + n_{2k+j}) \ .$$

On a 1-particle Bethe state with $x = e^{ip}$ these amplitudes combine as

$$x^{-1} - x^{-3} + x^{-5} - \ldots \to (x + x^{-1})^{-1}.$$

As a consequence, we find that on a general 1-particle Bethe state the operators H_p act diagonally with eigenvalue

$$E_p[x] = (x + x^{-1})^{2-2p} \ . \tag{30}$$

On a general f-particle state we find that $H_1 = 0$ for f even while $H_1 = 1$ on all states with f odd.

We have investigated the explicit form of $Q_p^\pm$ with $p \geq 2$ and H_q with $q \geq 3$ for small system sizes, confirming that operators satisfying the full algebra eq. (28) indeed exist. Restricting to 2-particle states, there is some freedom to choose the $Q_p^\pm$; what seems to be a canonical choice leads to eigenvalues

$$E_p[x, y] = (x + x^{-1})^{2-2p} + (-1)^p (y + y^{-1})^{2-2p} \tag{31}$$

on states with two exclusons. In addition, the H_p have an L-fold degenerate eigenvalue $E_p = 0$ when acting on states with a single Cooper pair. States with more particles show a similar pattern; for f particles with C Cooper pairs the eigenvalues E_p reduce to those of $f - 2C$ exclusons. The operators can be chosen such that on a state with exclusons at $\{x_a\}$, $a = 1, \ldots, f$, the eigenvalues of H_p with p even take the form

$$E_p[\{x_a\}] = \sum_{a=1}^{f} (x_a + x_a^{-1})^{2-2p} \qquad \text{for } p \text{ even.} \tag{32}$$

The algebraic structure in eq. (28) is highly reminiscent of the Yangian $Y(sl_2)$ which features as the symmetry algebra of various integrable models [21]. In the Haldane-Shastry spin-chain, the $SU(2)$-spin symmetry, with generators Q_0^A with $A = \pm, 3$, extends to a Yangian generated by Q_p^A, $p = 0, 1, \ldots$, with all Q_p^A commuting with conserved quantities H_q [12]. This algebraic structure beautifully reflects the underlying physical picture of 'spinons', with the Yangian generators $Q_p^\pm$ performing spin-flips on a multi-spinon state with all spins polarized. In our model here, the situation is similar, with the $Q_p^\pm$ sweeping out a full multiplet of eigenstates from a reference state with all particles 'polarized' on the same chain. Comparing with the Haldane-Shastry model, the role of the $SU(2)$-spin symmetry is taken over by supersymmetry, bringing the charges $Q_p^\pm$ and the conserved quantities H_q together in one algebraic structure. The actual form of the first nontrivial 'sYangian' generators $Q_1^\pm$ is reminiscent of the form of the (bosonic)

Yangian generators (sometimes referred to as the 'logarithmic Yangian') in integrable quantum field theories of massive particles and WZW models of conformal field theory [22].

The results of this section apply to open boundary conditions with the boundary magnetic field (3). An interesting open question is if this symmetry algebra can be deformed to apply to other boundary conditions, where the model is still solvable but there no longer exist the large degeneracies (10). If such a symmetry algebra does persist, it is then spectrum-generating like the Yangian.

We remark that in addition to all symmetries already mentioned, the model (with open boundary conditions) admits a discrete symmetry reminiscent of that of the Hubbard model at half filling. It is implemented by interchanging particles and holes *on the upper chain only*, leading to

$$(f_1, f_2) \leftrightarrow (L - f_1, f_2), \qquad E_0 \leftrightarrow 2L - 1 - E_0, \tag{33}$$

and in particular

$$N_2 \leftrightarrow C \; . \tag{34}$$

The latter relation shows that the particle-hole transformation maps the operators Q_p^+ (which each flip one particle from upper to the lower chain) into operators creating Cooper pairs.

5 Conclusion

We have discussed in depth a one-dimensional model with local interactions whose excitations are Cooper pairs and quasiparticles obeying exclusion statistics. We used the Bethe ansatz to derive these results, and found that various technical complications ended up being crucial to describing some very interesting physics.

Despite the presence of Cooper pairs, our system is not really a superconductor, because it is gapless. However, we believe it is possible to modify the Hamiltonian to give the exclusons a gap while preserving the Cooper pairs. The reason is the supersymmetry. If we deform the couplings without breaking the supersymmetry, the Witten index will not change [20], so the 2^L ground states with $E = 0$ will remain. Any supersymmetry-preserving deformation which gaps the exclusons will therefore leave the Cooper pairs. The next step would be to break the supersymmetry slightly so that Cooper pairs can have a kinetic energy below the gap.

Our model has some intriguing similarities to a solvable one-dimensional model of a superconductor. The "Russian-doll" model is an effective model of Cooper pairs, where time-reveral symmetry is broken by including a complex phase in the Hamiltonian [23]. This is very reminiscent of how time-reversal symmetry is broken in our model, and it would be interesting to find a deformation of our model whose Cooper pairs are described by the Russian-doll model.

We believe our model will prove useful in understanding exclusion statistics as well as superconductivity, since it can be treated with well-studied Bethe-ansatz methods. It should be possible to derive many more interesting properties for it. We are also hopeful that it may shed light on anyonic superconductivity. We find it amazing that 75 years after the invention of the Bethe ansatz, it is still being used to derive and explore fascinating properties of one-dimensional quantum systems.

We are very grateful to Fabian Essler for his patience. This work was supported in part by the foundations FOM and NWO of the Netherlands, and by NSF grant DMR-0412956.

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